Universal blocks of the *AdS*/*CFT* scattering matrix

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ABSTRACT: We identify certain blocks in the S-matrix describing the scattering of bound states of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring that allow for a representation in terms of universal R-matrices of Yangian doubles. For these cases, we use the formulas for Drinfeld's second realization of the Yangian in arbitrary bound-state representations to obtain the explicit expressions for the corresponding R-matrices. We then show that these expressions perfectly match with the previously obtained S-matrix blocks.

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Contents

1 Introduction

The study of integrable structures in the AdS/CFT correspondence [1–11] has now developed to a point where the exact solution of the planar spectral problem seems to be a concrete possibility [12–24]. A recent step in this direction was taken in [25], where the S-matrix describing the scattering of arbitrary bound states of the theory [26] has been obtained by employing the Yangian symmetry[1] [28–38]. In turn, the knowledge of the exact scattering matrix for generic states in the spectrum might be essential for better understanding the TBA approach [39].

The S-matrix found in [25] has a very interesting yet complicated structure. Understanding this structure could reveal a great deal of information on the underlying theory, by adopting the same ideology as for the case of the S-matrix for fundamental magnons. One immediate consequence of the result of [25] is that the S-matrix for arbitrary bound states is, by construction, given by a formula of the type

$$R = \Lambda^{op} \Lambda^{-1}$$

for a certain (super)matrix Λ. This naturally realizes the idea of factorizing (Drinfeld's) twists [40], and expresses the "triangular" factorization of the S-matrix.[2]

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[1]For interesting recent developments connecting Yangian symmetry and scattering amplitudes, see [27].
[2]We remark that, even in the case of the fundamental magnon S-matrix, this fact had not been explicitly shown before.

Given the overall difficulty of the problem, it is natural to start from reducing the S-matrix to smaller subsectors, and trying to reconduct them to well-understood mathematical objects. One expects the complete algebraic structure responsible for integrability to be some complicated and likely new type of quantum supergroup, whose properties have so far only appeared as pieces of a bigger puzzle. We intend to provide here another set of such pieces, coming from selected subsectors of the bound state S-matrix.

Our first focus will be on a particular subspace of states, which was the essential starting point of the construction in [25]. This is the space of bound states of the form (2.3) (Case I). The S-matrix transforms these states among themselves, with amplitudes controlled by a hypergeometric function (see (3.2) below). Since only the bosonic indices are allowed to be transformed, these states naturally host a representation of one of the $\mathfrak{su}(2)$ (the "bosonic" one) inside the (centrally-extended) $\mathfrak{psu}(2|2)$ Yangian, and it is a natural question to ask whether the S-matrix in this subspace is the representation of the universal R-matrix of the $\mathfrak{su}(2)$ Yangian double, $DY(\mathfrak{su}(2))$. We will show that this is indeed the case. This will be done by using the suitable evaluation representations for Drinfeld's second realization of $DY(\mathfrak{su}(2))$, originally obtained by [41], in the operator formalism of [42]. We will then obtain explicit expressions for the corresponding R-matrices in terms of hypergeometric functions.

We remark that the universal R-matrix and its realization in concrete representations is a well-studied subject in the mathematical literature, see e.g. [41, 43]. Here we will work out the expression for the universal R-matrix evaluated in generic finite-dimensional representations of $\mathfrak{su}(2)$ in our particular basis, suitable for comparison with the blocks of the bound-state scattering matrix. After completing the necessary steps, we will provide satisfactory evidence that these expressions match with our formula for the S-matrix (3.2) governing the scattering of Case I states. This matching ought also to be expected in view of the connection of the hypergeometric function in (3.2), shown to be related to a certain 6j-symbol in [25], to knot theory and solutions of the Yang-Baxter equation [44].

Moreover, one can notice, by analyzing the result of [25], how ubiquitous the amplitudes $\mathscr{X}_n^{k,l}$ (3.2) are in the S-matrix for both Case II and Case III, since by construction the latter is derived from the Case I amplitudes. These factors basically encode how the tail of bosonic states composing the bound states transform, with the fermions acting temporarily as spectators. The fact that we can interpret these fundamental building blocks as coming from the universal R-matrix of $\mathfrak{su}(2)$ suggests that the latter could be a genuine factor of the full universal R-matrix.

As a second task, we will focus on subspaces consisting of states with only one species of bosons and one species of fermions. These are other (four) subspaces, closed under the action of the S-matrix, and "transversal" to the Cases listed in [25]: namely, they contain vectors from Case I, II and III, as we will explain below. These subspaces host representations of $\mathfrak{gl}(1|1)$, and we will show that the S-matrix in these blocks can be obtained from the universal R-matrix of the $\mathfrak{gl}(1|1)$ Yangian double, $DY(\mathfrak{gl}(1|1))$. We will construct the general bound state representation for (Drinfeld's second realization of) $DY(\mathfrak{gl}(1|1))$, and then match with our results from [25]. The novelty with respect to the

$\mathfrak{su}(2)$ case is represented by the necessity of making an unusual choice of the evaluation parameter of the Yangian. This is an indication that these states, when scattering among themselves, respect an "effective" $\mathfrak{gl}(1|1)$ Yangian symmetry, whose parameters somehow encode the effect of the yet-unknown superior structure of the full universal R-matrix.

The paper is organized as follows. We will first summarize the structure and the main features of the bound state S-matrix constructed in [25]. We will also concisely review some notions concerning Drinfeld's first and second realization of the Yangian. We will then single out the $\mathfrak{su}(2)$ and the various $\mathfrak{gl}(1|1)$ subspaces, respectively. We will summarize the corresponding Yangian representations for arbitrary bound states, and explicitly evaluate their universal R-matrices, matching with our previously obtained results. We will conclude with an appendix, containing some computational details.

2 Structure of the bound state S-matrix

In this section, we report a summary of the results of [25], with the aim of fixing the notation and as a motivation to investigate the universal structure of the bound state scattering matrix.

We represent the algebra and the S-matrix action in the superspace formalism. One considers the vector space of analytic functions $\Phi(w, \theta)$ of two bosonic variables $w_{1,2}$ and two fermionic variables $\theta_{3,4}$. Any such function can be expanded as

$$\begin{aligned}
\Phi(w, \theta) &= \sum_{\ell=0}^{\infty} \Phi_\ell(w, \theta), \\
\Phi_\ell &= \phi^{a_1 \dots a_\ell} w_{a_1} \dots w_{a_\ell} + \phi^{a_1 \dots a_{\ell-1} \alpha} w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha + \\
&\quad \phi^{a_1 \dots a_{\ell-2} \alpha\beta} w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta.
\end{aligned} \tag{2.1}$$

The representation that describes ℓ-particle bound states is 4ℓ dimensional. It is realized on a graded vector space with basis $|e_{a_1 \dots a_\ell}\rangle, |e_{a_1 \dots a_{\ell-1}\alpha}\rangle, |e_{a_1 \dots a_{\ell-2}\alpha\beta}\rangle$, where a_i are bosonic indices and α, β are fermionic indices, and each of the basis vectors is totally symmetric in the bosonic indices and anti-symmetric in the fermionic indices. In terms of the above analytic functions, the basis vectors of the totally symmetric representation can be identified as $|e_{a_1 \dots a_\ell}\rangle \leftrightarrow w_{a_1} \dots w_{a_\ell}, |e_{a_1 \dots a_{\ell-1}\alpha}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-1}} \theta_\alpha$ and $|e_{a_1 \dots a_{\ell-1}\alpha\beta}\rangle \leftrightarrow w_{a_1} \dots w_{a_{\ell-2}} \theta_\alpha \theta_\beta$, respectively. In other words, one can find the atypical totally symmetric representation describing ℓ-particle bound states by restricting to terms Φ_ℓ. We refer to this basis of vectors as the "standard" basis.

We denote the bound state numbers of the scattering particles as ℓ_1 and ℓ_2, respectively. The tensor product of the corresponding bound state representations in superspace is given by:

$$\Phi_{\ell_1}(w, \theta) \Phi_{\ell_2}(v, \vartheta), \tag{2.2}$$

where w, θ denote the superspace variables of the first particle and v, ϑ describe the representation of the second particle. Because of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariance, when the S-matrix acts on such a (tensor-product) bound state representation space, it leaves five different subspaces invariant. Each of these subspaces is characterized by a specific assignment of

$\mathfrak{su}(2) \times \mathfrak{su}(2)$ Dynkin labels, which are quantum numbers that are trivially conserved during the scattering. Two pairs of these subspaces are simply related to each other by exchanging the type of fermions appearing, as described below. Therefore, only three non-equivalent cases are given, which we list here below.

Case I. The standard basis for this vector space, which we will concisely call V^{I}, is

$$|k,l\rangle^{\text{I}} \equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{\text{Space1}} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{\text{Space2}}, \tag{2.3}$$

for all $k + l = N$. The range of k, l here and in the cases below is straightforwardly read off from the definition of the states. In particular, in (2.3) k ranges from 0 to $\ell_1 - 1$, and l ranges from 0 to $\ell_2 - 1$. For fixed N, this gives in this case $N + 1$ different vectors. We get another copy of Case I if we exchange the index 3 with 4 in the fermionic variable, with the same S-matrix.

Case II. The standard basis for this space V^{II} is

$$|k,l\rangle_1^{\text{II}} \equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{} \ \underbrace{v_1^{\ell_2-l} v_2^l}_{},$$

$$|k,l\rangle_2^{\text{II}} \equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{},$$

$$|k,l\rangle_3^{\text{II}} \equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{},$$

$$|k,l\rangle_4^{\text{II}} \equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{}, \tag{2.4}$$

where $k + l = N$ as before.[3] It is easily seen that we get in this case $4N + 2$ states. Once again, exchanging 3 with 4 in the fermionic variable gives another copy of Case II, with the same S-matrix.

Case III. For fixed $N = k + l$, the dimension of this vector space V^{III} is $6N$. The standard basis is

$$|k,l\rangle_1^{\text{III}} \equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{} \ \underbrace{v_1^{\ell_2-l} v_2^l}_{},$$

$$|k,l\rangle_2^{\text{III}} \equiv \underbrace{w_1^{\ell_1-k} w_2^k}_{} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{},$$

$$|k,l\rangle_3^{\text{III}} \equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{} \ \underbrace{v_1^{\ell_2-l} v_2^l}_{},$$

$$|k,l\rangle_4^{\text{III}} \equiv \underbrace{\theta_3 \theta_4 w_1^{\ell_1-k-1} w_2^{k-1}}_{} \ \underbrace{\vartheta_3 \vartheta_4 v_1^{\ell_2-l-1} v_2^{l-1}}_{},$$

$$|k,l\rangle_5^{\text{III}} \equiv \underbrace{\theta_3 w_1^{\ell_1-k-1} w_2^k}_{} \ \underbrace{\vartheta_4 v_1^{\ell_2-l} v_2^{l-1}}_{},$$

$$|k,l\rangle_6^{\text{III}} \equiv \underbrace{\theta_4 w_1^{\ell_1-k} w_2^{k-1}}_{} \ \underbrace{\vartheta_3 v_1^{\ell_2-l-1} v_2^l}_{}. \tag{2.5}$$

The different cases are mapped into one another by use of the (opposite) coproducts of the (Yangian) symmetry generators.

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[3]We will from now on, with no risk of confusion, omit indicating "Space 1" and "Space 2" under the curly brackets.

The S-matrix has the following block-diagonal form:

$$
\mathbb{S} = \begin{pmatrix} \boxed{\mathscr{X}} & & & & \\ & \boxed{\mathscr{Y}} & & \mbox{\Large 0} & \\ & & \boxed{\mathscr{Z}} & & \\ & \mbox{\Large 0} & & \boxed{\mathscr{Y}} & \\ & & & & \boxed{\mathscr{X}} \end{pmatrix}.
\tag{2.6}
$$

The outer blocks scatter states from V^{I}

$$
\mathscr{X} : V^{\mathrm{I}} \longrightarrow V^{\mathrm{I}}
\tag{2.7}
$$

$$
|k,l\rangle^{\mathrm{I}} \; \mapsto \; \sum_{m=0}^{k+l} \mathscr{X}_m^{k,l} |m, k+l-m\rangle^{\mathrm{I}},
\tag{2.8}
$$

where $\mathscr{X}_m^{k,l}$ is given by eq. (4.11) in [25]. We will report its explicit expression in the next section, formula (3.2). The blocks $\mathscr{Y}$ describe the scattering of states from V^{II}

$$
\mathscr{Y} : V^{\mathrm{II}} \longrightarrow V^{\mathrm{II}}
\tag{2.9}
$$

$$
|k,l\rangle_j^{\mathrm{II}} \; \mapsto \; \sum_{m=0}^{k+l} \sum_{j=1}^{4} \mathscr{Y}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{II}}.
\tag{2.10}
$$

These S-matrix elements are given in eq. (5.18) of [25], but we will not need their explicit expression here. Finally, the middle block deals with the third case

$$
\mathscr{Z} : V^{\mathrm{III}} \longrightarrow V^{\mathrm{III}}
\tag{2.11}
$$

$$
|k,l\rangle_j^{\mathrm{III}} \; \mapsto \; \sum_{m=0}^{k+l} \sum_{j=1}^{6} \mathscr{Z}_{m;i}^{k,l;j} |m, k+l-m\rangle_j^{\mathrm{III}},
\tag{2.12}
$$

with $\mathscr{Z}_{m;i}^{k,l;j}$ from eq. (6.11) in [25]. Similarly, these expressions are not needed for the sake of the present discussion, and we refer to [25] for their details.

We recall that the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ string bound state S-matrix is then obtained by taking two copies of the above S-matrix, and multiplying the result by the *square* of the following phase factor [42, 45]:

$$
S_0(p_1, p_2) = \left(\frac{x_1^-}{x_1^+}\right)^{\frac{\ell_2}{2}} \left(\frac{x_2^+}{x_2^-}\right)^{\frac{\ell_1}{2}} \sigma(x_1, x_2) \sqrt{G(\ell_2 - \ell_1) G(\ell_2 + \ell_1)} \prod_{q=1}^{\ell_1 - 1} G(\ell_2 - \ell_1 + 2q), \tag{2.13}
$$

where, in our conventions,

$$
G(Q) = \frac{u_1 - u_2 + \frac{Q}{2}}{u_1 - u_2 - \frac{Q}{2}}.
\tag{2.14}
$$

Here, u is given in the standard variables by

$$
u \equiv \frac{g}{4i} \left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} \right).
\tag{2.15}
$$

3   Invariant subspaces

We will describe here in detail the two type of subspaces of states we will be focussing our attention on.

3.1   The $\mathfrak{su}(2)$ subspace

The first subspace is given by states belonging to Case I in the above classification. We remind that the $\mathfrak{psu}(2|2)$ algebra has two ("bosonic" and "fermionic", according to the indices they transform) $\mathfrak{su}(2)$ subalgebras, with generators $\mathbb{L}_b^a$ and $\mathbb{R}_\beta^\alpha$, respectively. The first ones satisfy the following commutation relations:

$$[\mathbb{L}_a^{\ b}, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c,$$
$$[\mathbb{L}_a^{\ b}, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c. \tag{3.1}$$

The states from Case I form a natural representation on which the "bosonic" $\mathfrak{su}(2)$ sub-algebra of $\mathbb{L}_b^a$'s acts. The latter transforms the bosonic variables, and leaves the only two (equal-type) fermions presents as spectators. Furthermore, the Case I S-matrix satisfies the Yang-Baxter equation by itself, and it is of difference form up to the overall scalar factor $\mathcal{D}$. This means that such S-matrix should naturally come from the universal R-matrix of the $\mathfrak{su}(2)$ Yangian double [41]. The Case I S-matrix is given by [25]

$$\mathscr{X}_n^{k,l} = (-1)^{k+n}\, \pi\mathcal{D}\frac{\sin[(k-\ell_1)\pi]\,\Gamma(l+1)}{\sin[\ell_1\pi]\sin[(k+l-\ell_2-n)\pi]\,\Gamma(l-\ell_2+1)\Gamma(n+1)} \tag{3.2}$$
$$\times \frac{\Gamma(n+1-\ell_1)\Gamma\left(l+\frac{\ell_1-\ell_2}{2}-n-\delta u\right)\Gamma\left(1-\frac{\ell_1+\ell_2}{2}-\delta u\right)}{\Gamma\left(k+l-\frac{\ell_1+\ell_2}{2}-\delta u+1\right)\Gamma\left(\frac{\ell_1-\ell_2}{2}-\delta u\right)}$$
$$\times {}_4\tilde{F}_3\left(-k, -n, \delta u+1-\frac{\ell_1-\ell_2}{2}, \frac{\ell_2-\ell_1}{2}-\delta u; 1-\ell_1, \ell_2-k-l, l-n+1; 1\right),$$

where one has defined ${}_4\tilde{F}_3(x,y,z,t;r,v,w;\tau) = {}_4F_3(x,y,z,t;r,v,w;\tau)/[\Gamma(r)\Gamma(v)\Gamma(w)]$ and

$$\mathcal{D} = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}}. \tag{3.3}$$

The quantity δu equals $u_1 - u_2$, with u given by (2.15). One can check that this S-matrix satisfies the YBE, and we will indeed show that this formula coincides with what one obtains from the Yangian universal R-matrix [41], with the same evaluation parameter u (2.15).

3.2   The $\mathfrak{su}(1|1)$ subspace

The other subspace we will consider is obtained by restricting the bound states to having bosonic and fermionic indices of only one respective type. For definiteness, we will take the bosonic index to be 1 and the fermionic index to be 3. There are four copies of this subspace, corresponding to the four different choices of these indices we can make. The embedding of this subspace in the full bound state representation is spanned by the vectors

$$\left\{|0,0\rangle_1^{\text{III}},\ |0,0\rangle_1^{\text{II}},\ |0,0\rangle_2^{\text{II}},\ |0,0\rangle^{\text{I}}\right\}. \tag{3.4}$$

As one can see, this subspace takes particular states from all three Cases listed above, yet being closed under the action of the S-matrix. This means that the S-matrix for this subsector corresponds to a block-diagonal 4×4 matrix. The two Case II states mix with an S-matrix given by formula (3.15) in [25]. The amplitude for the Case III state present is normalized to 1, while for the Case I state it is simply the factor $\mathcal{D}$ (3.3). Putting this together, one obtains

$$
\mathbb{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\frac{p_2}{2}} \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & \frac{\sqrt{\ell_1}\eta(p_1)}{\sqrt{\ell_2}\eta(p_2)} \frac{x_2^+ - x_2^-}{x_1^+ - x_2^-} & 0 \\ 0 & \frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}} \frac{\sqrt{\ell_2}\eta(p_2)}{\sqrt{\ell_1}\eta(p_1)} \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-} & e^{i\frac{p_1}{2}} \frac{x_1^- - x_2^-}{x_1^+ - x_2^-} & 0 \\ 0 & 0 & 0 & \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{e^{i\frac{p_1}{2}}}{e^{i\frac{p_2}{2}}} \end{pmatrix}. \tag{3.5}
$$

We remark that, taken in the fundamental representation, and suitably un-twisted in order to eliminate the braiding factors coming from the nontrivial coproduct [46–48], this matrix coincides with the S-matrix of [49]. It is readily checked that this matrix satisfies the Yang-Baxter equation by itself, therefore it is natural to ask whether it is the representation of a known (Yangian) universal R-matrix.

In the remainder of the paper we will discuss the universal R-matrices for the Yangian doubles associated to $\mathfrak{su}(2)$ and $\mathfrak{gl}(1|1)$, and show that they coincide with the above discussed bound state S-matrix blocks [25]. The construction relies on Drinfeld's second realization of the Yangian, which we will review in the next section.

4 Drinfeld's realizations of the Yangian

In this section we report, for convenience of the reader, the defining relations of the Yangian of a simple Lie algebra $\mathfrak{g}$, in Drinfeld's first and second realization.[4] For a thorough treatment of the subject, the reader is referred for instance to [44, 53–55].

The first realization [56] is obtained as follows. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra generated by $\mathfrak{J}^A$ with commutation relations $\left[\mathfrak{J}^A, \mathfrak{J}^B\right] = f_C^{AB}\mathfrak{J}^C$, and equipped with a non-degenerate invariant bilinear form. The Yangian is the infinite-dimensional (Hopf) algebra generated by level zero generators $\mathfrak{J}^a$ and level one generators $\widehat{\mathfrak{J}}^a$

$$
\left[\mathfrak{J}^A, \mathfrak{J}^B\right] = f_C^{AB}\mathfrak{J}^C, \tag{4.1}
$$

$$
\left[\mathfrak{J}^A, \widehat{\mathfrak{J}}^B\right] = f_C^{AB}\widehat{\mathfrak{J}}^C, \tag{4.2}
$$

subject to certain (Serre-relations type of) constraints.

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[4]When superalgebras will be involved, all formulas will be understood in their natural graded generalization (see for example [34, 50–52]). The non-simplicity of $\mathfrak{gl}(1|1)$ will not be an obstacle, as its Yangian satisfies a similar set of defining relations.

The second realization [57] is given in terms of generators $\kappa_{i,m}, \xi_{i,m}^{\pm}$, $i = 1, \ldots, \text{rank}\,\mathfrak{g}$, $m = 0, 1, 2, \ldots$ and relations

$$[\kappa_{i,m}, \kappa_{j,n}] = 0, \qquad\qquad \left[\kappa_{i,0}, \xi_{j,m}^{+}\right] = a_{ij}\,\xi_{j,m}^{+}, \tag{4.3}$$

$$\left[\kappa_{i,0}, \xi_{j,m}^{-}\right] = -a_{ij}\,\xi_{j,m}^{-}, \qquad\qquad \left[\xi_{j,m}^{+}, \xi_{j,n}^{-}\right] = \delta_{i,j}\,\kappa_{j,n+m},$$

$$\left[\kappa_{i,m+1}, \xi_{j,n}^{+}\right] - \left[\kappa_{i,m}, \xi_{j,n+1}^{+}\right] = \frac{1}{2}a_{ij}\{\kappa_{i,m}, \xi_{j,n}^{+}\},$$

$$\left[\kappa_{i,m+1}, \xi_{j,n}^{-}\right] - \left[\kappa_{i,m}, \xi_{j,n+1}^{-}\right] = -\frac{1}{2}a_{ij}\{\kappa_{i,m}, \xi_{j,n}^{-}\},$$

$$\left[\xi_{i,m+1}^{\pm}, \xi_{j,n}^{\pm}\right] - \left[\xi_{i,m}^{\pm}, \xi_{j,n+1}^{\pm}\right] = \pm\frac{1}{2}a_{ij}\{\xi_{i,m}^{\pm}, \xi_{j,n}^{\pm}\},$$

$$i \neq j, \qquad n_{ij} = 1 + |a_{ij}|, \;\; Sym_{\{k\}}\left[\xi_{i,k_1}^{\pm}, \left[\xi_{i,k_2}^{\pm}, \ldots \left[\xi_{i,k_{n_{ij}}}^{\pm}, \xi_{j,l}^{\pm}\right] \cdots \right]\right] = 0.$$

In these formulas, a_{ij} is the (symmetric) Cartan matrix.

Drinfeld [57] gave the isomorphism between the two realizations as follows. Let us define a Chevalley-Serre basis for $\mathfrak{g}$ as composed of Cartan generators $\mathfrak{H}_i$, and positive (negative) simple roots $\mathfrak{E}_i^{+}$ ($\mathfrak{E}_i^{-}$, respectively). Also, let us define the corresponding Cartan-Weyl basis for $\mathfrak{g}$, composed of generators $\mathfrak{H}_i$ and $E_{\beta}^{\pm}$. One has then

$$\begin{array}{lll}
\kappa_{i,0} = \mathfrak{H}_i, & \xi_{i,0}^{+} = \mathfrak{E}_i^{+}, & \xi_{i,0}^{-} = \mathfrak{E}_i^{-}, \\
\kappa_{i,1} = \hat{\mathfrak{H}}_i - v_i, & \xi_{i,1}^{+} = \hat{\mathfrak{E}}_i^{+} - w_i, & \xi_{i,1}^{-} = \hat{\mathfrak{E}}_i^{-} - z_i,
\end{array} \tag{4.4}$$

where

$$v_i = \frac{1}{4}\sum_{\beta \in \Delta^{+}} (\alpha_i, \beta)\,(E_{\beta}^{-}E_{\beta}^{+} + E_{\beta}^{+}E_{\beta}^{-}) - \frac{1}{2}H_i^2, \tag{4.5}$$

$$w_i = \frac{1}{4}\sum_{\beta \in \Delta^{+}} \left(E_{\beta}^{-}\,\mathrm{ad}_{\mathfrak{E}_i^{+}}(E_{\beta}^{+}) + \mathrm{ad}_{\mathfrak{E}_i^{+}}(E_{\beta}^{+})E_{\beta}^{-}\right) - \frac{1}{4}\{\mathfrak{E}_i^{+}, \mathfrak{H}_i\}, \tag{4.6}$$

$$z_i = \frac{1}{4}\sum_{\beta \in \Delta^{+}} \left(\mathrm{ad}_{E_{\beta}^{-}}(\mathfrak{E}_i^{-})E_{\beta}^{+} + E_{\beta}^{+}\,\mathrm{ad}_{E_{\beta}^{-}}(\mathfrak{E}_i^{-})\right) - \frac{1}{4}\{\mathfrak{E}_i^{-}, \mathfrak{H}_i\}. \tag{4.7}$$

Here Δ^{+} denotes the set of positive root vectors, and the adjoint action is defined as $\mathrm{ad}_x(y) = [x, y]$.

The double of the Yangian admits a universal R-matrix which endows it with a quasi-triangular structure. Explicit formulas have been given in [41] by making use of Drinfeld's second realization. In the general case the expressions are rather complicated, therefore we will not report them here. We will instead report in what follows the concrete examples relevant to our subspaces of interest.

5 Universal R-matrix for $\mathfrak{su}(2)$

We will now proceed to compute the universal R-matrix for the $\mathfrak{su}(2)$ block of our bound state S-matrix, following [41]. The derivation is split up into three parts, corresponding to the factorization

$$R = R_E R_H R_F, \tag{5.1}$$

R_E and R_F being "root" factors, while R_H is a purely diagonal "Cartan" factor. As we mentioned in the above, one works in Drinfeld's second realization of the Yangian. The map (4.4) between the first and the second realization becomes in this case

$$h_0 = h, \qquad e_0 = e, \qquad f_0 = f,$$
$$h_1 = \hat{h} - v, \qquad e_1 = \hat{e} - w, \qquad f_1 = \hat{f} - z, \qquad (5.2)$$

where

$$v = \frac{1}{2}(\{f, e\} - h^2), \qquad w = -\frac{1}{4}\{e, h\}, \qquad z = -\frac{1}{4}\{f, h\}. \qquad (5.3)$$

The first realization is given by

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$
$$[\hat{h}, e] = [h, \hat{e}] = 2e, \qquad [\hat{h}, f] = [h, \hat{f}] = -2f, \qquad [\hat{e}, f] = [e, \hat{f}] = h, \qquad (5.4)$$

and in evaluation representation one has $\hat{h} = uh = u(w_2\partial_{w_2} - w_1\partial_{w_1})$, $\hat{e} = ue = u(w_2\partial_{w_1})$ and $\hat{f} = uf = u(w_1\partial_{w_2})$. By applying Drinfeld's map (5.2) to this evaluation representation, one first finds the level 0 and 1 generators of the second realization. For generic bound state representations they depend on second order derivatives. After some manipulations, one can put the level 1 generators in a simple form, very suggestive of the possible generalization at level n. This form reads

$$f_n = f\left(u + \frac{h-1}{2}\right)^n, \qquad (5.5)$$

$$e_n = e\left(u + \frac{h+1}{2}\right)^n, \qquad (5.6)$$

$$h_n = ef_n - fe_n. \qquad (5.7)$$

These generators coincide with what obtained in [41] for generic highest-weight representations of $Y(\mathfrak{su}(2))$.

It is easy to check that these generators satisfy the correct defining relations obtained by specializing (4.3):

$$[h_m, h_n] = 0, \qquad\qquad [e_m, f_n] = h_{n+m},$$
$$[h_0, e_m] = 2e_m, \qquad\qquad [h_0, f_m] = -2f_m,$$
$$[h_{m+1}, e_n] - [h_m, e_{n+1}] = \{h_m, e_n\},$$
$$[h_{m+1}, f_n] - [h_m, f_{n+1}] = -\{h_m, f_n\},$$
$$[e_{m+1}, e_n] - [e_m, e_{n+1}] = \{e_m, e_n\},$$
$$[f_{m+1}, f_n] - [f_m, f_{n+1}] = -\{f_m, f_n\}. \qquad (5.8)$$

The universal R-matrix for the double of the Yangian of $\mathfrak{sl}(2)$ reads

$$R = R_E R_H R_F, \qquad (5.9)$$

where

$$R_E = \overrightarrow{\prod_{n \geq 0}} \exp(-e_n \otimes f_{-n-1}), \tag{5.10}$$

$$R_F = \overleftarrow{\prod_{n \geq 0}} \exp(-f_n \otimes e_{-n-1}), \tag{5.11}$$

$$R_H = \prod_{n \geq 0} \exp\left\{ \mathrm{Res}_{u=v} \left[\frac{\mathrm{d}}{\mathrm{d}u} (\log H^+(u)) \otimes \log H^-(v + 2n + 1) \right] \right\}. \tag{5.12}$$

One has defined

$$\mathrm{Res}_{u=v}\left(A(u) \otimes B(v)\right) = \sum_k a_k \otimes b_{-k-1} \tag{5.13}$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$, and the so-called Drinfeld's currents are given by

$$E^\pm(u) = \pm \sum_{\substack{n \geq 0 \\ n < 0}} e_n u^{-n-1}, \qquad\qquad F^\pm(u) = \pm \sum_{\substack{n \geq 0 \\ n < 0}} f_n u^{-n-1}$$

$$H^\pm(u) = 1 \pm \sum_{\substack{n \geq 0 \\ n < 0}} h_n u^{-n-1} . \tag{5.14}$$

The arrows on the products indicate the ordering one has to follow in the multiplication, and are a consequence of the normal ordering prescription for the root factors in the universal R-matrix [41]. For the generic bound state representations which we have described above, the ordering will be essential to get the correct result, and cannot be ignored as it accidentally happens for the case of the fundamental representation of $\mathfrak{su}(2)$. We will review the computation of the three relevant factors of the universal R-matrix in appendix A, and report here only the final results in our conventions.

We define the state $\langle A, B \rangle \langle C, D \rangle$ as made of an A number of w_1's, a B number of w_2's in the first space, and analogously C and D for v_1, v_2 in the second space. We also define

$$\begin{aligned}
c_i &= u_1 - \frac{A - B + 1}{2} - i, \\
d_i &= u_2 - \frac{C - D - 1}{2} + i, \\
\tilde{c}_i &= u_2 - \frac{C - D + 1}{2} - i, \\
\tilde{d}_i &= u_1 - \frac{A - B - 1}{2} + i, \tag{5.15}
\end{aligned}$$

and

$$\delta u = u_1 - u_2.$$

One has for the factor R_F

$$\overleftarrow{\prod_{n \geq 0}} \exp[-f_n \otimes e_{-1-n}]|k, l\rangle = \sum_m A_m(A, B, C, D) |k - m, l + m\rangle, \tag{5.16}$$

with

$$A_m(A, B, C, D) = m! \binom{B}{m}\binom{C}{m} \prod_{i=0}^{m-1} \frac{1}{c_0 - d_0 - i - m + 1}. \tag{5.17}$$

The Cartan part is then given by

$$R_H \langle A, B \rangle \ \langle C, D \rangle = \frac{2^{1-2\delta u} \pi \ \Gamma\left(\frac{2\delta u+A+B+C-D+2}{2}\right) \Gamma\left(\frac{2\delta u+B-A+C+D+2}{2}\right)}{\Gamma(\frac{\delta u-A+B+C-D}{2})\Gamma(\frac{\delta u-A+B+C-D+2}{2})\Gamma(\frac{2\delta u-A-B-C-D}{4})}$$

$$\times \frac{\Gamma\left(\frac{2\delta u-A+B-C-D}{2}\right) \ \Gamma\left(\frac{2\delta u-A-B+C-D}{2}\right)}{\Gamma(\frac{2\delta u+A+B-C-D+2}{4})\Gamma(\frac{2\delta u-A-B+C+D+2}{4})\Gamma(\frac{2\delta u+A+B+C+D+4}{4})} \langle A, B\rangle\langle C, D\rangle$$

$$\equiv \mathcal{H}(A, B, C, D) \ \langle A, B\rangle\langle C, D\rangle. \tag{5.18}$$

Finally, the factor R_E is given by

$$\overrightarrow{\prod_{n\geq 0}} \exp[-e_n \otimes f_{-1-n}]|k, l\rangle = \sum_m B_m(A, B, C, D)|k+m, l-m\rangle, \tag{5.19}$$

where

$$B_m(A, B, C, D) = m! \binom{A}{m}\binom{D}{m} \prod_{i=0}^{m-1} \frac{1}{\tilde{d}_0 - \tilde{c}_0 - i + m - 1}. \tag{5.20}$$

We are now ready to put things together and evaluate the action of the universal R-matrix of $\mathfrak{su}(2)$ on Case I states. From formulas (5.16), (5.18) and (5.19), we obtain

$$R|k, l\rangle = \sum_{m=0}^{min(B,C)} \sum_{n=0}^{min(A,D)+m} B_n(A+m, B-m, C-m, D+m) \tag{5.21}$$

$$\times \mathcal{H}(A+m, B-m, C-m, D+m) A_m(A, B, C, D) |k-m+n, l+m-n\rangle,$$

where

$$A = \ell_1 - k - 1, \qquad\qquad B = k,$$
$$C = \ell_2 - l - 1, \qquad\qquad D = l, \tag{5.22}$$

and the various factors are given by formulas (5.17), (5.18) and (5.20). It is now easy to convert the above expression into

$$R|k, l\rangle = \sum_{n=0}^{k+l} R_n |n, k+l-n\rangle. \tag{5.23}$$

In order to find the amplitudes R_n, we proceed as follows. Taking into account the presence of binomial factors in the expressions for A_m and B_n, which naturally truncate the sum when m, n lie outside the correct intervals, we can extend the summation indices to run from $-\infty$ to ∞. In this way, manipulations of the above sums are easier, and one ends up with

$$R_n = \sum_{m=-n+k}^{\infty} A_m(\ell_1 - k - 1, k, \ell_2 - l - 1, l)$$

$$\times \mathcal{H}(\ell_1 - k - 1 + m, k - m, \ell_2 - l - 1 - m, l + m)$$

$$\times B_{n-k+m}(\ell_1 - k - 1 + m, k - m, \ell_2 - l - 1 - m, l + m). \tag{5.24}$$

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The result of the summation can be obtained by restriction to the suitable integer values of the parameters of the following meromorphic function, expressed in terms of hypergeometric functions:

$$
\begin{aligned}
R_n = a_1 \big[\, {}_6F_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6; \beta_1, \beta_2, \beta_3, \beta_4, \beta_5; 1) + y \\
\times {}_6F_5(\alpha_1 - 1, \alpha_2 - 1, \alpha_3 - 1, \alpha_4 - 1, \alpha_5 - 1, \alpha_6 - 1; \beta_1 - 1, \beta_2 - 1, \beta_3 - 1, \beta_4 - 1, \beta_5 - 1; 1) \big],
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= 2 + N - n, & \alpha_2 &= 1 - n, & \alpha_3 &= 1 + \ell_1 - n, \\
\alpha_4 &= 2 + N - n - \ell_2, & \alpha_5 &= 1 + N - 2n - \delta u + \frac{\ell_1 - \ell_2}{2}, & \alpha_6 &= 1 + l - n - \delta u + \frac{\ell_1 - \ell_2}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_1 &= \alpha_1 - l, & \beta_2 &= 2 + N - \frac{\ell_1 + \ell_2}{2} - n - \delta u, \\
\beta_3 &= 1 + \frac{\ell_1 - \ell_2}{2} - n - \delta u, & \beta_4 &= 2 + N - n - \delta u + \frac{\ell_1 - \ell_2}{2}, & \beta_5 &= 1 + \frac{\ell_1 + \ell_2}{2} - n - \delta u.
\end{aligned}
$$

We have also defined

$$
y = \frac{(k-n+1)(2N-\ell_1-\ell_2-2n-2\delta u+2)(2N+\ell_1-\ell_2-2(n+\delta u-1))\big((\ell_1-2(n+\delta u))^2-\ell_2^2\big)}{16(N-n+1)(N-\ell_2-n+1)n(n-\ell_1)(2l+\ell_1-\ell_2-2(n+\delta u))}, \tag{5.25}
$$

and

$$
\begin{aligned}
a_1 = -&\frac{(-1)^{k-n}\pi \sin((n+1)\pi)(N-n+1)(N-\ell_2-n+1)(n-\ell_1)(2(N-2n-\delta u)+\ell_1-\ell_2)(2(n-l+\delta u)-\ell_1+\ell_2)}{(k-n+1)(2(N-n-\delta u+1)-\ell_1-\ell_2)(2(N-n-\delta u+1)+\ell_1-\ell_2)} \\
\times &\frac{\sin\left(\frac{\pi(2(N-2n-\delta u-2)+\ell_1-\ell_2)}{4}\right)\sin^2\left(\frac{\pi(2(N-2n-\delta u-1)+\ell_1-\ell_2)}{4}\right)\sin\left(\frac{\pi(2(N-2n-\delta u)+\ell_1-\ell_2)}{4}\right)}{\sin\left(\frac{\pi(2(n+\delta u+1)-\ell_1+\ell_2)}{2}\right)\sin\left(\frac{\pi(2(n-N+\delta u)-\ell_1+\ell_2)}{2}\right)\sin\left(\frac{\pi(2(n-N+\delta u)+\ell_1+\ell_2)}{2}\right)\sin\left(\frac{\pi(\ell_1+\ell_2-2(n+\delta u+1))}{2}\right)} \\
\times &\frac{\Gamma(k+1)\Gamma(\ell_2-l)\Gamma(1-n)\Gamma(N-\ell_2-n+1)\Gamma\left(N+\frac{\ell_1-\ell_2}{2}-2n-\delta u\right)\Gamma\left(l+\frac{\ell_1-\ell_2}{2}-n-\delta u\right)}{\Gamma(k-n+1)\Gamma\left(N-\frac{\ell_1+\ell_2}{2}-n-\delta u+1\right)\Gamma\left(N+\frac{\ell_1-\ell_2}{2}-n-\delta u+1\right)\Gamma\left(\frac{2\delta u+2-\ell_1-\ell_2}{4}\right)\Gamma\left(\frac{2\delta u+4-\ell_1-\ell_2}{4}\right)} \\
\times &\frac{4^{2-\delta u}\mathcal{D}\sin((n-N+\ell_2)\pi)}{\Gamma\left(\frac{\ell_1+\ell_2+2\delta u}{4}\right)\Gamma\left(\frac{\ell_1+\ell_2+2\delta u+2}{4}\right)\Gamma\left(\frac{\ell_1-\ell_2-2(n+\delta u-1)}{2}\right)\Gamma\left(\frac{\ell_1+\ell_2-2(n+\delta u-1)}{2}\right)}. \tag{5.26}
\end{aligned}
$$

The quantity N is again here equal to $k + l$.

We have checked that this coincides with the r.h.s. of (3.2) for a large selection of choices of the integer parameters, keeping δu arbitrary, when taking into account the proper normalization. We have in fact, with the notations of [25],

$$
\mathcal{D} \frac{\Gamma\left(\frac{1}{4}(2 + \ell_1 - \ell_2 + 2\delta u)\right)\Gamma\left(\frac{1}{4}(2 + \ell_2 - \ell_1 + 2\delta u)\right)}{\Gamma\left(\frac{1}{4}(4 - \ell_1 - \ell_2 + 2\delta u)\right)\Gamma\left(\frac{1}{4}(\ell_1 + \ell_2 + 2\delta u)\right)} R_n = \mathscr{X}_n^{k,l}. \tag{5.27}
$$

The ratio of gamma functions appearing in the above formula is the (inverse of the) so-called "character" of the universal R-matrix in evaluation representations [41], namely its action on states of highest-weight $\lambda_i = l_i - 1$.

As a remark, we notice that the formulas for the coproducts of the generators of $DY(\mathfrak{su}(2))$ (as well as for $DY(\mathfrak{gl}(1|1))$ which will be studied next) are explicitly known at arbitrary level n in Drinfeld's second realization. It is easy to see that the coproducts of the level $n = 1$ generators discussed in this section coincide with the truncation to $\mathfrak{su}(2)$ of the general expressions obtained in [34]. This indicates how this smaller Yangian we have been discussing here can be embedded in the $\mathfrak{psu}(2|2)$ one.

6 Universal R-matrix for $\mathfrak{gl}(1|1)$

In this section, we focus on four other subsectors of the entire bound state representation space, closed under the action of the S-matrix. We will show that the S-matrix block(s) scattering these sectors can be obtained from the universal R-matrix of a Yangian double, in suitable evaluation representations.

Each of these sectors is obtained by considering bound states made of only one type of boson and one type of fermion. The algebra transforming the states inside these sectors is an $\mathfrak{sl}(1|1)$. As it is known, this type of superalgebras (with a degenerate Cartan matrix) do not admit a universal R-matrix, therefore we will introduce an extra Cartan generator [58] and study the Yangian of the algebra $\mathfrak{gl}(1|1)$ instead.[5] Let us start with the canonical derivation, and adapt the representation later in order to exactly match with our S-matrix.

We will follow [41, 59]. The super Yangian double $DY(gl(1|1))$ is the Hopf algebra generated by the elements e_n, f_n, h_n, k_n, with n an integer number, satisfying (Drinfeld's second realization)

$$
\begin{aligned}
&[h_m\,,\,h_n] = [h_m\,,\,k_n] = [k_m\,,\,k_n] = 0, \\
&[k_m\,,\,e_n] = [k_m\,,\,f_n] = 0, \\
&[h_0\,,\,e_n] = -2e_n\,, \qquad\qquad\qquad [h_0\,,\,f_n] = 2f_n, \\
&[h_{m+1}\,,\,e_n] - [h_m\,,\,e_{n+1}] + \{h_m\,,\,e_n\} = 0, \\
&[h_{m+1}\,,\,f_n] - [h_m\,,\,f_{n+1}] - \{h_m\,,\,f_n\} = 0, \\
&\{e_m\,,\,e_n\} = \{f_m\,,\,f_n\} = 0, \\
&\{e_m\,,\,f_n\} = -k_{m+n}.
\end{aligned}
\tag{6.1}
$$

Drinfeld's currents are given by

$$
E^{\pm}(t) = \pm \sum_{\substack{n\geq 0 \\ n<0}} e_n t^{-n-1}\,, \qquad\qquad F^{\pm}(t) = \pm \sum_{\substack{n\geq 0 \\ n<0}} f_n t^{-n-1}, \tag{6.2}
$$

$$
H^{\pm}(t) = 1 \pm \sum_{\substack{n\geq 0 \\ n<0}} h_n t^{-n-1}\,, \qquad\qquad K^{\pm}(t) = 1 \pm \sum_{\substack{n\geq 0 \\ n<0}} k_n t^{-n-1}. \tag{6.3}
$$

The universal R-matrix reads

$$
\mathcal{R} = \mathcal{R}_+ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_-, \tag{6.4}
$$

where

$$
\mathcal{R}_+ = \overrightarrow{\prod_{n\geq 0}} \exp(-e_n \otimes f_{-n-1}), \tag{6.5}
$$

$$
\mathcal{R}_- = \overleftarrow{\prod_{n\geq 0}} \exp(f_n \otimes e_{-n-1}), \tag{6.6}
$$

---

[5]For the purposes of the universal R-matrix, it will not make any difference to consider real forms of the algebras when needed.

$$\mathcal{R}_1 = \prod_{n \geq 0} \exp \left\{ \mathrm{Res}_{t=z} \left[(-1) \frac{d}{dt} (\log H^+(t)) \otimes \ln K^-(z + 2n + 1) \right] \right\}, \tag{6.7}$$

$$\mathcal{R}_2 = \prod_{n \geq 0} \exp \left\{ \mathrm{Res}_{t=z} \left[(-1) \frac{d}{dt} (\log K^+(t)) \otimes \ln H^-(z + 2n + 1) \right] \right\}, \tag{6.8}$$

and again

$$\mathrm{Res}_{t=z} \left(A(t) \otimes B(z) \right) = \sum_k a_k \otimes b_{-k-1} \tag{6.9}$$

for $A(t) = \sum_k a_k t^{-k-1}$, $B(z) = \sum_k b_k z^{-k-1}$.

One can show that the following bound state representation, acting on monomials made of a generic bosonic state v and a generic fermionic state θ, satisfies all the defining relations of the second realization (6.1):

$$e_n = \lambda^n \, a \, \theta \partial_v, \qquad\qquad f_n = \lambda^n \, d \, v \partial_\theta,$$
$$k_n = -\lambda^n \, ad \, (v \partial_v + \theta \partial_\theta), \qquad h_n = (\lambda + \ell - 1)^n (v \partial_v - \theta \partial_\theta). \tag{6.10}$$

As usual, we denote by ℓ the number of components of the bound state. At this stage, a and d are arbitrarily chosen representation labels, and λ is a generic spectral parameter independent of a, d. We will later specify the values they have to take in order to match with the bound state S-matrix in these subsectors. Let us start by selecting w_1 as our boson v, and θ_3 as our fermion θ. Let us also define a basis of this first subsector in the following way:

$$\left\{ |0, 0\rangle_1^{\mathrm{III}}, \ |0, 0\rangle_1^{\mathrm{II}}, \ |0, 0\rangle_2^{\mathrm{II}}, \ |0, 0\rangle^{\mathrm{I}} \right\}. \tag{6.11}$$

We first compute

$$\mathcal{R}_- = \overleftarrow{\prod_{n \geq 0}} \exp[f_n \otimes e_{-n-1}] \tag{6.12}$$

in our bound state representation. Because of the fermionic nature of the operators $f_n \otimes e_{-n-1}$, the above expression simplifies to

$$\begin{aligned}
\mathcal{R}_- &= 1 + \sum_{n \geq 0} f_n \otimes e_{-n-1} \\
&= 1 + \sum_{n \geq 0} \frac{u_1^n}{u_2^{n+1}} f \otimes e \\
&= 1 - \frac{f \otimes e}{\delta \lambda}
\end{aligned} \tag{6.13}$$

Considering that this term will act non-trivially only on a state with a fermion in the first space,[6] we easily obtain

$$\mathcal{R}_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{a_2 d_1 \ell_2}{\delta \lambda} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.14}$$

---

[6]Tensor products of generators act according to the rule $(X \otimes Y)|a\rangle \otimes |b\rangle = (-)^{[Y][a]} X|a\rangle \otimes Y|b\rangle$, where $[x]$ denotes the fermionic grading of x.

We have defined

$$\delta\lambda = \lambda_1 - \lambda_2. \tag{6.15}$$

Similarly, one finds

$$\mathcal{R}_+ = 1 - \sum_{n \geq 0} e_n \otimes f_{-n-1}$$
$$= 1 + \frac{e \otimes f}{\delta\lambda}, \tag{6.16}$$

which, written in matrix form, looks like

$$\mathcal{R}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{a_1 d_2 \ell_1}{\delta\lambda} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.17}$$

Let us now turn to the Cartan part. For this, we first need to compute the currents. They are found to be

$$H^{\pm} = 1 + \frac{h}{1 + \lambda - \ell - t}, \tag{6.18}$$

$$K^{\pm} = 1 + \frac{k}{\lambda - t}, \tag{6.19}$$

where we used the fact that both h and k are diagonal operators. In appropriate domains one then has in particular

$$-\frac{d}{dt} \log H^+ = \sum_{m=1}^{\infty} \left\{ (\lambda + \ell - 1)^m - (\lambda + \ell - 1 - h)^m \right\} t^{-m-1} \tag{6.20}$$

and

$$\log K^-(z + 2n + 1) = \log K^-(2n + 1) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{(\lambda - 1 - 2n)^m} - \frac{1}{(\lambda - 1 - 2n - k)^m} \right\} \frac{z^m}{m}. \tag{6.21}$$

Straightforwardly computing the residue and performing the sum yields, in matrix form,

$$\mathcal{R}_1 = \frac{\Gamma\left(\frac{\delta\lambda + \ell_1}{2}\right) \Gamma\left(\frac{\delta\lambda - a_2 d_2 \ell_2}{2}\right)}{\Gamma\left(\frac{\delta\lambda}{2}\right) \Gamma\left(\frac{\delta\lambda + \ell_1 - a_2 d_2 \ell_2}{2}\right)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\delta\lambda - a_2 d_2 \ell_2}{\delta\lambda} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda - a_2 d_2 \ell_2}{\delta\lambda} \end{pmatrix}. \tag{6.22}$$

One can perform an analogous derivation for R_2 and find

$$\mathcal{R}_2 = \frac{\Gamma\left(\frac{\delta\lambda + a_1 d_1 \ell_1 + 2}{2}\right) \Gamma\left(\frac{\delta\lambda - \ell_2 + 2}{2}\right)}{\Gamma\left(\frac{\delta\lambda + 2}{2}\right) \Gamma\left(\frac{\delta\lambda + a_1 d_1 \ell_1 - \ell_2 + 2}{2}\right)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\delta\lambda}{\delta\lambda + a_1 d_1 \ell_1} & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda}{\delta\lambda + a_1 d_1 \ell_1} \end{pmatrix}. \tag{6.23}$$

Multiplying everything out finally gives us the universal R-matrix in our bound state representation:

$$\mathcal{R} = A \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{a_2 d_2 \ell_2}{\delta\lambda + a_1 d_1 \ell_1} & \frac{a_1 d_2 \ell_1}{\delta\lambda + a_1 d_1 \ell_1} & 0 \\ 0 & \frac{a_2 d_1 \ell_2}{\delta\lambda + a_1 d_1 \ell_1} & \frac{\delta\lambda}{\delta\lambda + a_1 d_1 \ell_1} & 0 \\ 0 & 0 & 0 & \frac{\delta\lambda - a_2 d_2 \ell_2}{\delta\lambda + a_1 d_1 \ell_1} \end{pmatrix}, \tag{6.24}$$

where

$$A = \frac{\Gamma\left(\frac{\delta\lambda + \ell_1}{2}\right) \Gamma\left(\frac{\delta\lambda + a_1 d_1 \ell_1 + 2}{2}\right) \Gamma\left(\frac{\delta\lambda - \ell_2 + 2}{2}\right) \Gamma\left(\frac{\delta\lambda - a_2 d_2 \ell_2}{2}\right)}{\Gamma\left(\frac{\delta\lambda}{2}\right) \Gamma\left(\frac{\delta\lambda + 2}{2}\right) \Gamma\left(\frac{\delta\lambda + a_1 d_1 \ell_1 - \ell_2 + 2}{2}\right) \Gamma\left(\frac{\delta\lambda + \ell_1 - a_2 d_2 \ell_2}{2}\right)}. \tag{6.25}$$

For $a_i = d_i = \ell_i = 1$ this reduces to the formula in [59],

$$\mathcal{R} \propto \mathbb{1} + \frac{P}{\delta\lambda}, \tag{6.26}$$

where P is the graded permutation matrix.

But we can also take a, d to be the representation labels of the supercharges in the centrally extended $\mathfrak{psu}(2|2)$ superalgebra, i.e.

$$a = \sqrt{\frac{g}{2\ell}}\eta, \qquad d = \sqrt{\frac{g}{2\ell}}\frac{x^+ - x^-}{i\eta}. \tag{6.27}$$

This corresponds to considering the generators e, f as the restriction to this subsector of the two supercharges $\mathbb{Q}_1^3$ and $\mathbb{G}_3^1$. It is now readily seen that by choosing λ to be $\frac{g}{2i}x^-$, we can exactly reproduce[7] the 4×4 block (3.5) from (6.24), after we properly normalize it and introduce the appropriate braiding factors. To normalize, we simply divide the formula coming from the universal R-matrix by A (6.25). To introduce the braiding factors, we need to twist it by [42]

$$U_2^{-1}(p_1)\,\mathcal{R}\,U_1(p_2),$$

with $\mathrm{U}(p) = diag(1, e^{-ip/2})$.

There is also another choice for a, d from the $\mathfrak{psu}(2|2)$ algebra. Namely, one can also restrict the supercharges $\mathbb{Q}_2^4$ and $\mathbb{G}_4^2$ to this sector. This means that our parameters a, d will now become the c, b from the bound state representation

$$b = \sqrt{\frac{g}{2\ell}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right), \qquad c = -\sqrt{\frac{g}{2\ell}}\frac{\eta}{\zeta x^+}. \tag{6.28}$$

Remarkably, in order to match with (3.5), one has to choose $\lambda = \frac{ig}{2x^-}$ and $\zeta_1 = \zeta_2$. The correct braiding factors can be incorporated by means of the inverse of the above mentioned twist [42].

A similar argument can finally be seen to hold for all the other subsectors corresponding to different fixed bosonic and fermionic indices.

While it is likely that in the full universal R-matrix (where one is supposed to have at once all generators of $\mathfrak{psu}(2|2)$) some kind of "average" of the two situations will occur,[8]

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[7]This is similar to the observation in [49] for the the case of the fundamental representation.

[8]In the fundamental representation, this is exemplified by some of the formulas in [60].

we have shown here that the S-matrix in these subspaces can be "effectively" described by the universal R-matrix of $DY(\mathfrak{gl}(1|1))$ taken in (two inequivalent choices of) evaluation representations.

Acknowledgments

A Revisiting the $Y(\mathfrak{su}(2))$ computation

In this appendix, we give the computational details for the $\mathfrak{su}(2)$ case.

A.1 The factor R_F

Let us first compute how R_F acts on an arbitrary Case I state. We find

$$\overset{\leftarrow}{\prod_{n \geq 0}} \exp\left[-f_n \otimes e_{-1-n}\right]|k,l\rangle = \sum_m A_m |k-m, l+m\rangle. \tag{A.1}$$

The term A_m is built up out of m copies of $-f \otimes e$ acting on the state $\langle A,B\rangle\langle C,D\rangle$, which is made of an A number of w_1's, a B number of w_2's in the first space, and analogously C and D for v_1, v_2 in the second space. In view of (A.1), we find that such terms can come from different exponentials, i.e. with different n's, or from the same exponential. One first needs to know how the product of m f's acts on the state $\langle A,B\rangle$.

We conveniently define as in the main text

$$c_i = u_1 - \frac{A-B+1}{2} - i, \tag{A.2}$$

$$d_i = u_2 - \frac{C-D-1}{2} + i. \tag{A.3}$$

In general one has

$$\begin{aligned}
f_{n_m}\ldots f_{n_2}f_{n_1}\langle A,B\rangle &= f_{n_m}\ldots f\left(u+\frac{h-1}{2}\right)^{n_2} f\left(u+\frac{h-1}{2}\right)^{n_1}\langle A,B\rangle \\
&= f_{n_m}\ldots f\left(u+\frac{h-1}{2}\right)^{n_2} f\,(c_0)^{n_1}\langle A,B\rangle \\
&= B\,(c_0)^{n_1} f_{n_m}\ldots f\left(u+\frac{h-1}{2}\right)^{n_2}\langle A+1,B-1\rangle \\
&= B(B-1)\,(c_0)^{n_1}\,(c_1)^{n_2} f_{n_m}\ldots f_{n_3}\langle A+2,B-2\rangle \\
&= \frac{B!}{(B-m)!}c_0^{n_1}\ldots c_{m-1}^{n_m}\langle A+m,B-m\rangle.
\end{aligned} \tag{A.4}$$

Similar expressions hold for e_n acting on $\langle C, D \rangle$, but with d_i instead of c_i, and producing the state $\langle C - m, D + m \rangle$. When we consider terms like this coming from the ordered exponential (A.1), we always have that $n_i \geq n_{i-1}$. In case $n_i = n_{i+1}$, we also pick up a combinatorial factor coming from the series of the exponential. Putting all of this together, we find

$$A_m = (-)^m \frac{B!}{(B-m)!} \frac{C!}{(C-m)!} \left\{ \sum_{n_1 \leq \ldots \leq n_m} \frac{1}{N(\{n_1, \ldots, n_m\})} \frac{c_0^{n_1}}{d_0^{n_1+1}} \cdots \frac{c_{m-1}^{n_m}}{d_{m-1}^{n_m+1}} \right\},$$

$$N(\{n_1, \ldots, n_m\}) = \frac{1}{\mathrm{ord} S(\{n_1, \ldots, n_m\})}. \tag{A.5}$$

N is a combinatorial factor which is defined as the inverse of the order of the permutation group of the set $\{n_1, \ldots, n_m\}$. For example, $N(\{1, 1, 2\}) = \frac{1}{2}$ and $N(\{1, 1, 1, 2, 3, 3, 4, 5\}) = \frac{1}{3!} \frac{1}{2!} = \frac{1}{12}$. By using the fact that $c_i = c_{i+1} + 1, d_i = d_{i+1} - 1$, one can evaluate this sum explicitly and find

$$A_m(A, B, C, D) = m! \binom{B}{m} \binom{C}{m} \prod_{i=0}^{m-1} \frac{1}{c_0 - d_0 - i - m + 1}, \tag{A.6}$$

where we have indicated the dependence on the parameters A, B, C, D of the state we are acting on. As one can easily see using (5.15), the resulting expression is manifestly of difference form.

A.2 The factor R_H

Next is the Cartan part. First, we work out

$$h_n \langle A, B \rangle = \left\{ (A+1)B \left[u - \frac{A - B + 1}{2} \right]^n - (B+1)A \left[u - \frac{A - B - 1}{2} \right]^n \right\} \langle A, B \rangle.$$

We then recall the definition of $H_{\pm}$ from (5.14). From the explicit realization we give in the main text it follows that

$$H_+(t)\langle A, B \rangle = H_-(t)\langle A, B \rangle = \left\{ 1 - \frac{(A+1)B}{u - t - \frac{1}{2}(A - B + 1)} + \frac{A(B+1)}{u - t - \frac{1}{2}(AB - 1)} \right\} \langle A, B \rangle.$$

Defining $K_{\pm} = \log H_{\pm}$, the Cartan part of the universal R-matrix can be written as

$$R_H = \prod_{n \geq 0} \exp \left[\mathrm{Res}_{t=x} \left(\frac{d}{dt} K_+(t) \otimes K_-(x + 2n + 1) \right) \right], \tag{A.7}$$

where the residue is defined in (5.13). We have to find the suitable series representations corresponding to $\frac{d}{dt} K_+(t)$ and $K_-(x + 2n + 1)$. With an appropriate choice of domains for the variables t and x, one can write in particular

$$\frac{d}{dt} K_+(t) = \sum_{m \geq 1} \{\alpha_1^m + \alpha_2^m - \alpha_3^m - \alpha_4^m\} t^{-m-1}, \tag{A.8}$$

$$K_-(x + 2n + 1) = K_-(0) + \sum_{m \geq 1} \{\beta_1^{-m} + \beta_2^{-m} - \beta_3^{-m} - \beta_4^{-m}\} \frac{x^m}{m}, \tag{A.9}$$

where

$$\alpha_1 = u_1 + \frac{1}{2}(A+B+1), \qquad \alpha_2 = u_1 - \frac{1}{2}(A+B+1),$$

$$\alpha_3 = u_1 - \frac{1}{2}(A-B+1), \qquad \alpha_4 = u_1 - \frac{1}{2}(A-B-1), \qquad (A.10)$$

and

$$\beta_1 = u_2 - 2n + \frac{1}{2}(D-C-1), \qquad \beta_2 = u_2 - 2n + \frac{1}{2}(D-C-3),$$

$$\beta_3 = u_2 - 2n + \frac{1}{2}(D+C-1), \qquad \beta_4 = u_2 - 2n - \frac{1}{2}(D+C+3), \qquad (A.11)$$

This leads to

$$R_H \langle A,B \rangle \langle C,D \rangle = \frac{2^{1-2\delta u} \pi \; \Gamma\left(\frac{2\delta u + A+B+C-D+2}{2}\right) \Gamma\left(\frac{2\delta u + B-A+C+D+2}{2}\right)}{\Gamma\left(\frac{\delta u - A+B+C-D}{2}\right)\Gamma\left(\frac{\delta u - A+B+C-D+2}{2}\right)\Gamma\left(\frac{2\delta u - A-B-C-D}{4}\right)} \times$$

$$\times \frac{\Gamma\left(\frac{2\delta u - A+B-C-D}{2}\right) \; \Gamma\left(\frac{2\delta u - A-B+C-D}{2}\right)}{\Gamma\left(\frac{2\delta u + A+B-C-D+2}{4}\right)\Gamma\left(\frac{2\delta u - A-B-C+D+2}{4}\right)\Gamma\left(\frac{2\delta u + A+B+C+D+4}{4}\right)} \langle A,B \rangle \langle C,D \rangle$$

$$\equiv \mathcal{H}(A,B,C,D) \langle A,B \rangle \langle C,D \rangle, \qquad (A.12)$$

where

$$\delta u = u_1 - u_2.$$

A.3 The factor R_E

We will now compute R_E. One has

$$\overrightarrow{\prod_{n \geq 0}} \exp[-e_n \otimes f_{-1-n}]|k,l\rangle = \sum_m B_m |k+m, l-m\rangle. \qquad (A.13)$$

Let us define as in the main text

$$\tilde{c}_i = u_2 - \frac{C-D+1}{2} - i, \qquad (A.14)$$

$$\tilde{d}_i = u_1 - \frac{A-B-1}{2} + i. \qquad (A.15)$$

The term B_m is this time built up out of m copies of $-e \otimes f$ acting on the state $\langle A,B \rangle \langle C,D \rangle$. One has

$$e_{n_m} \dots e_{n_2} e_{n_1} \langle A,B \rangle = e_{n_m} \dots e \left(u + \frac{h+1}{2} \right)^{n_2} e \left(u + \frac{h+1}{2} \right)^{n_1} \langle A,B \rangle$$

$$= e_{n_m} \dots e \left(u + \frac{h+1}{2} \right)^{n_2} e \left(\tilde{d}_0 \right)^{n_1} \langle A,B \rangle$$

$$= A \left(\tilde{d}_0 \right)^{n_1} e_{n_m} \dots e \left(u + \frac{h+1}{2} \right)^{n_2} \langle A-1, B+1 \rangle$$

$$= A(A-1) \left(\tilde{d}_0 \right)^{n_1} \left(\tilde{d}_1 \right)^{n_2} e_{n_m} \dots e_{n_3} \langle A-2, B+2 \rangle$$

$$= \frac{A!}{(A-m)!} \tilde{d}_0^{n_1} \dots \tilde{d}_{m-1}^{n_m} \langle A-m, B+m \rangle. \qquad (A.16)$$

Similar expressions hold for f_n acting on $\langle C, D \rangle$, with $\tilde{c}_i$ instead of $\tilde{d}_i$, and producing the state $\langle C + m, D - m \rangle$. From the ordered exponential (A.13) we have now $n_i \leq n_{i-1}$. In case $n_i = n_{i+1}$, we again pick up the same combinatorial factor as in the calculation of R_F, coming from the series of the exponential. Putting all of this together, we find

$$B_m = \frac{A!}{(A-m)!} \frac{D!}{(D-m)!} \left\{ \sum_{n_1 \geq \ldots \geq n_m} \frac{1}{N(\{n_1, \ldots, n_m\})} \frac{\tilde{d}_0^{n_1}}{\tilde{c}_0^{n_1+1}} \cdots \frac{\tilde{d}_{m-1}^{n_m}}{\tilde{c}_{m-1}^{n_m+1}} \right\},$$

$$N(\{n_1, \ldots, n_m\}) = \frac{1}{\text{ord} S(\{n_1, \ldots, n_m\})}, \tag{A.17}$$

where N is defined as in the formulas for R_F. The sum evaluates at

$$B_m(A, B, C, D) = m! \binom{A}{m} \binom{D}{m} \prod_{i=0}^{m-1} \frac{1}{\tilde{d}_0 - \tilde{c}_0 - i + m - 1}. \tag{A.18}$$

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